Non-invertible symmetry, and string tensions beyond N-ality

Yuya Tanizaki (Yukawa institute, Kyoto)

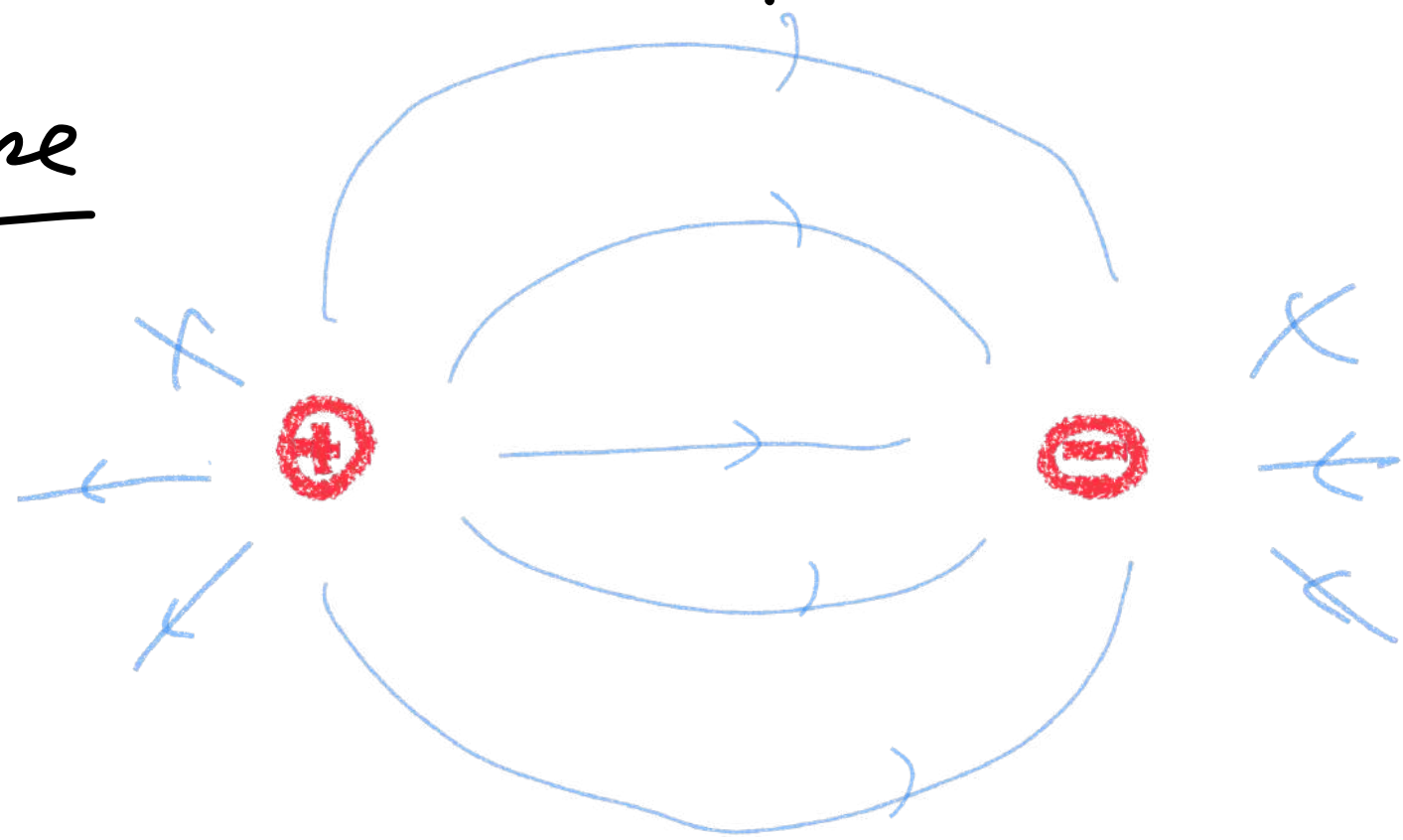Mendel Nguyen, Mithat Ünsal (North Carolina State U.)

Motivation : Confining strings in SU(N) YM

In order to characterise "confinement" in gauge theories, we look at the interparticle potential for test charges.
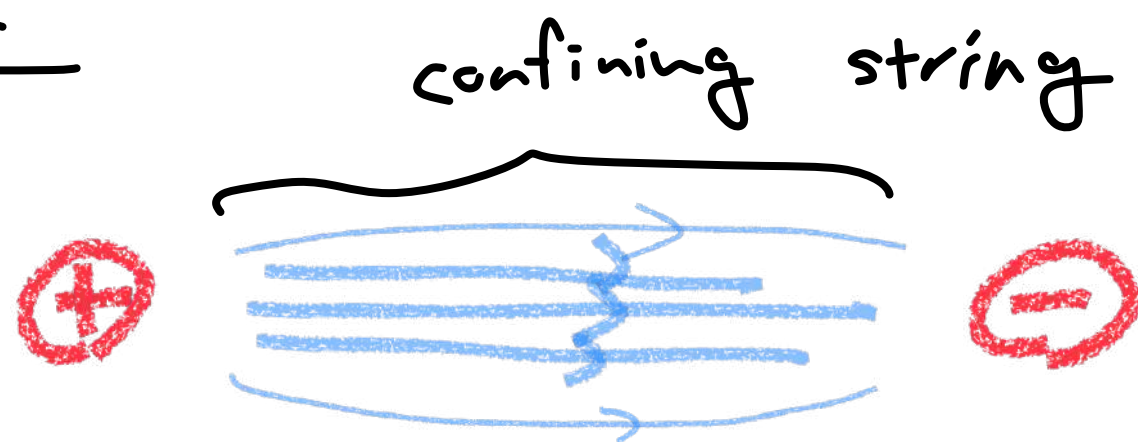
Coulomb phase

Potentials

$$V(r) \propto \frac{1}{r^{D-2}} \qquad (D: \text{spatial dim.})$$

Higgs phase

dynamical charges

$$V(r) \propto \text{const.}$$

Confining phase

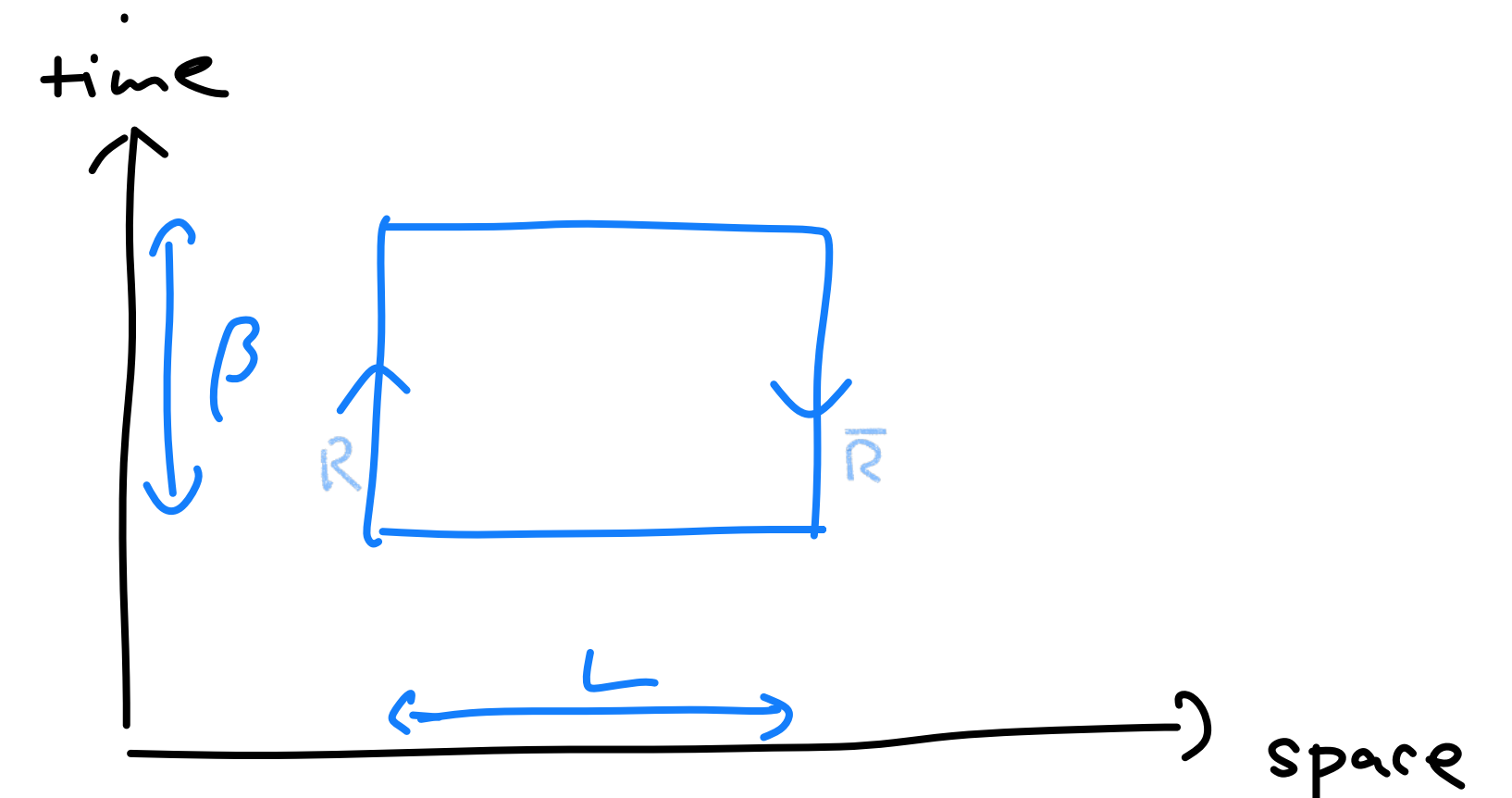confining string

$$V(r) \propto r.$$

Wilson Loop & Area law

In order to detect the interparticle potential in QFT, we introduce a loop operator, such as the Wilson loop.

$$W_R(C) = \text{tr}_R\left[\mathcal{P} e^{i \oint_C a_\mu dx^\mu} \right]$$

$$\rightsquigarrow \langle W_R(C) \rangle \sim \exp\left(- \beta\, V_R(L)\right),$$

where $V_R(L)$ is a potential for charges $R, \overline{R}$, separated by L.

Especially, when the system is confining

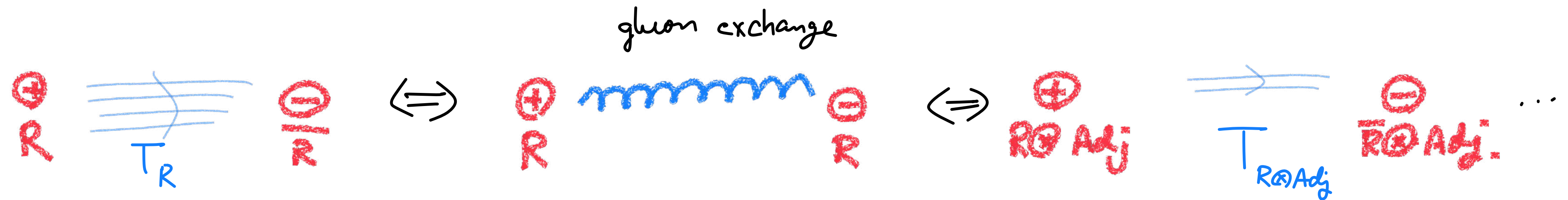$$\langle W_R(C) \rangle \sim \exp\left(- T_R \times \text{Area}(C)\right).$$

string tension

Spectrum of confining strings : N-ality

T_R (string tension) are important quantities characterizing confinement.

$\Rightarrow$ How do T_R's depend on the gauge representation R ?

N-ality σ_R depends only on $\#$(boxes of Young tab.) mod N.

(Believed to be true for pure $SU(N)$ YM for 3 & 4 dim.)

gluon exchange

$$\overset{\oplus}{R} \; \overline{\underset{R}{\ominus}} \quad \Longleftrightarrow \quad \overset{\oplus}{R} \;\text{mmmmm}\; \overset{\ominus}{R} \quad \Longleftrightarrow \quad \overset{\oplus}{R \otimes Adj} \quad T_{R \otimes Adj} \quad \overline{\overset{\ominus}{R \otimes Adj.}} \quad \dots$$

T_R

1-form symmetry (center symmetry)

In modern understandings, for relativistic QFTs, $\left(\frac{cf.}{\text{Gaiotto, Kapustin, Seiberg, Willet '14}}\right)$

Symmetry $\overset{def}{:=}$ existence of "topological" operators $U(M_{d-p-1})$.

Continuous sym. of $\mathcal{L}$ $\longrightarrow$ Noether current $j^{\mu}_{\alpha} = \delta_{\alpha}\phi \frac{\partial \mathcal{L}}{\partial(\partial_{r}\phi)}$.

$$\implies U_{\varepsilon}(M_{d-1}) := \exp\left(i\int_{M_{d-1}} \varepsilon_{\alpha}\, j^{\mu_1}_{\alpha}\, \varepsilon_{r_1 \mu_2 \cdots \mu_d}\, dx^{\mu_2}\cdots dx^{\mu_d}\right).$$

$\mathbb{Z}_N$ 1-form sym.

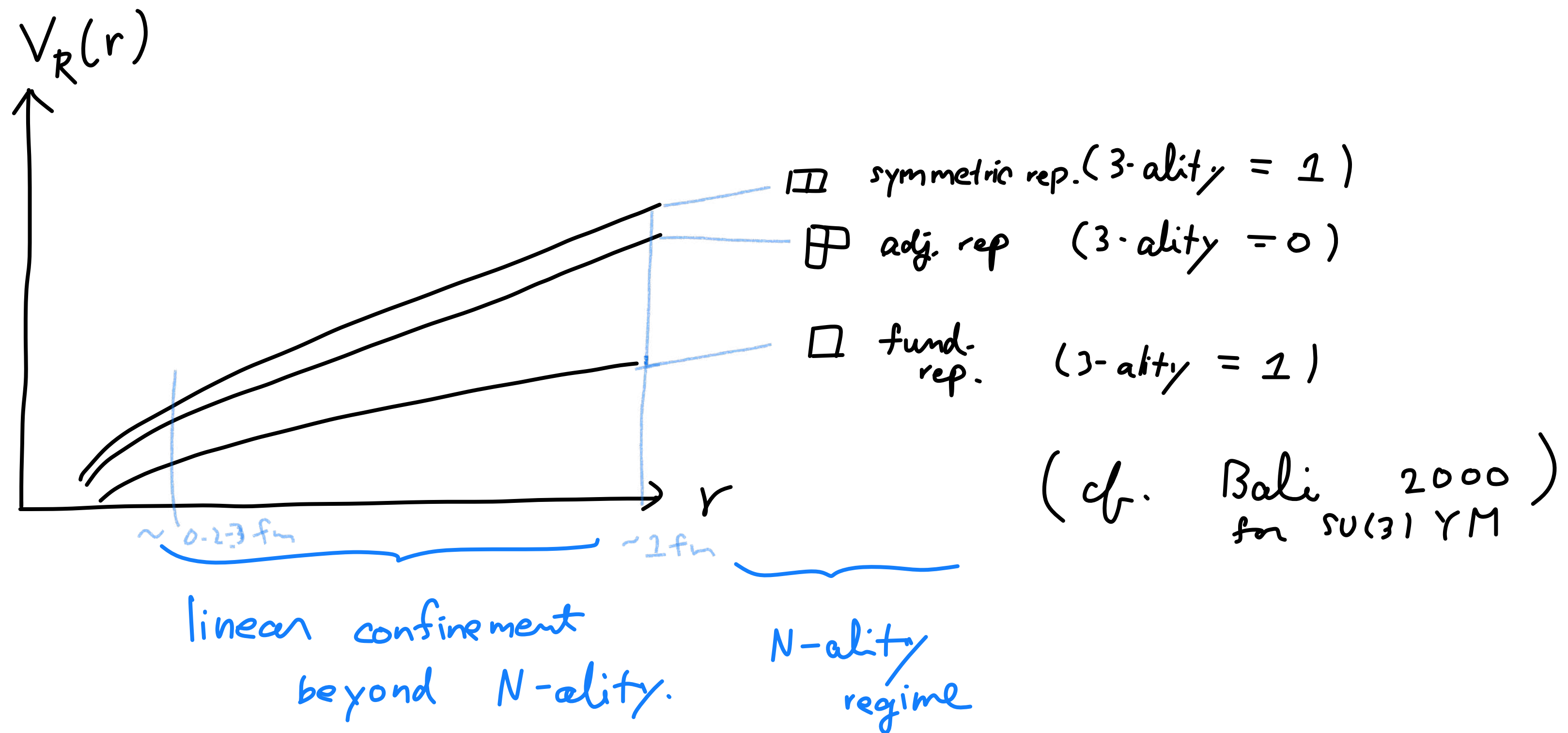For $SU(N)$ YM, there are codim-2 topological operator $U(M_{d-2})$.

[Gukov-Witten operator]

$$W_R(C) \longmapsto e^{\frac{2\pi i}{N}\cdot |R|} W_R(C). \quad \Rightarrow \text{Nice explanation of N-ality rule.}$$

Beyond N-ality

N-ality is an important feature in the deep IR regime.

However, it does not kick in immediately after the linear confinement occurs.

$V_R(r)$

□□ symmetric rep. (3-ality $= 1$)

adj. rep (3-ality $= 0$)

□ fund. rep. (3-ality $= 1$)

(cf. Bali 2000) for SU(3) YM

~0.2-3 fm ~ 1 fm

linear confinement beyond N-ality.

N-ality regime

Is there some "nice" way to understand the beyond N-ality regime?

What is this talk about?

We want to get better understandings on confining strings beyond N-ality.

① Construction of a toy model (3d semi-Abelian gauge theory)

- $G_{gauge} = U(1)^{N-1} \rtimes S_N$
- $\mathbb{Z}_N$ 1-form sym.

② Studying its string tensions (string tensions beyond N-ality)

- $T_{Adj} \simeq 2 T_{fd.} \neq 0$

③ "Symmetry" explanation on beyond N-ality (non-invertible sym. in 3d)

✱ Disclaimer : This talk does not contain SU(N) YM at all, but it's anyway interesting !!

$(2+1)d$ $U(1)^{N-1}$ gauge model with S_N global sym

Gauge field $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$ with $tr(\vec{a}) = \sum_{i=1}^{N} a_i = 0$.

$\left(\text{Using simple roots } \alpha_i \text{ of } SU(N), \text{ we can rewrite it as} \right.$
$$\vec{a}_{(x)} = \sum_{i=1}^{N-1} \tilde{a}_i(x) \, \alpha_i \left. \right)$$

The Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \vec{f}_{\mu\nu} \cdot \vec{f}_{\mu\nu}$$

is symmetric under the global S_N permutation

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} \longmapsto \begin{pmatrix} a_{\sigma(1)} \\ a_{\sigma(2)} \\ \vdots \\ a_{\sigma(N)} \end{pmatrix} \qquad (\sigma \in S_N).$$

$\left(\text{* Here, I write down the continuum Lagrangian,} \right.$

$\left. \text{but, in the actual computations, we used the lattice model.} \right)$

Monopoles

This (lattice) model acquires the mass gap due to monopole gas.

As we put the theory on lattice, Bianchi id.

$$d f = d (da) = 0$$

is **violated:** $df \neq 0$. (lattice monopoles)

Abelian duality : $*\vec{f} = d\vec{\sigma}$ w/ $2\pi\mu$ – periodic scalars $\vec{\sigma}$.

Monopoles can be described as $e^{i\alpha \cdot \vec{\sigma}(x)}$.

$$\mathcal{L}_{eff} = \frac{g^2}{2\pi} \int \left\{ |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \cdot \sum_{\substack{\alpha: \text{positive} \\ \text{roots}}} (1 - \cos(\alpha \cdot \vec{\sigma})) \right\} d^3x$$

$\Rightarrow$ mass gap

Some remarks

Idea is very similar to the Polyakov model:

$$3d \quad SU(N) \quad YM + Adj. \text{ scalar } \Phi$$

$$\xrightarrow{\text{Higgsing by } \langle \Phi \rangle} \quad 3d \quad U(1)^{N-1} \text{ gauge theory } + \text{ monopoles.}$$

$$\mathcal{L}_{eff} \sim |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \sum_{\alpha: \text{ positive simple roots}} (1 - \cos(\alpha \cdot \vec{\sigma}))$$

The Polyakov model, however, does **not** realize S_N symmetry at low-energies, while <u>our model does.</u>

<u>Our model</u>

S_N ✓

mass

(N-1) dual photons have degenerate mass

<u>Polyakov model</u>

S_N ✗

mass

(N-1) dual photons have different masses.

Gauging S_N : $U(1)^{N-1} \rtimes S_N$ gauge theory

As our $U(1)^{N-1}$ model has a manifest S_N symmetry,
let us gauge it!

$\Rightarrow$ Physical operators must be local S_N invariant.

Wilson formulation

On the lattice, we can realize it by saying that

$$\underset{\ell}{\overset{\text{link variable}}{U}} = \underbrace{P_\ell}_{\substack{N \times N \text{ matrix} \\ \text{for } S_N \text{ perm.}}} \cdot \underbrace{C_\ell}_{= \begin{pmatrix} e^{i a_1} & & \\ & \ddots & \\ & & e^{i a_N} \end{pmatrix} \in U(1)^{N-1}}$$

$$S = \beta_1 \underbrace{\operatorname{tr}\left(\mathbb{1}_N - \prod_{\ell \in \partial P} (P_\ell \cdot C_\ell) \right)}_{U(1)^{N-1} \rtimes S_N \text{ plaquette action}} + \beta_2 \underbrace{\operatorname{tr}\left(\mathbb{1}_N - \prod_{\ell \in \partial P} P_\ell \right)}_{S_N \text{ plaquette action}}.$$

Taking the limit $\beta_2 \to +\infty$, we impose the flatness condition

$$\prod_{\ell \in \partial p} P_\ell = \mathbb{1}_N .$$

$\Rightarrow$ S_N-gauge fields are topological.

In this limit, the local dynamics does not change under the gauging of S_N:

$$\left\langle \mathcal{O}_1(x_1) \underset{\nwarrow \ S_N \text{ Wilson line}}{W_{S_N}(x_1, x_2)} \mathcal{O}_2(x_2) \right\rangle_{U(1)^{N-1} \rtimes S_N}$$

$$= \left\langle \mathcal{O}_1(x_1) \ \mathcal{O}_2(x_2) \right\rangle_{U(1)^{N-1}} .$$

S_N-non-singlet operators.

- Thanks to this nature, we can peform explicit computations as the theory is essentially Abelian.

- Symmetry is affected, however.

1-form symmetries of semi-Abelian theory

Gauge invariance

$$U(1)^{N-1}$$

$$\xrightarrow{\quad S_N\text{-gauging}\quad}$$

$$U(1)^{N-1} \rtimes S_N \quad \text{"semi-Abelian"}$$
$$(\subset SU(N) : \text{subgrp of } SU(N))$$

Center of the gauge group

$$Z(U(1)^{N-1}) = U(1)^{N-1}$$

$$\Downarrow$$

$$U(1)^{N-1} \quad 1\text{-form sym.}$$

$$Z(U(1)^{N-1} \rtimes S_N) = \mathbb{Z}_N.$$

$$\Downarrow$$

$$\boxed{\mathbb{Z}_N \quad 1\text{-form sym.}}$$

⇝ Infinitely many string tensions can be expected.
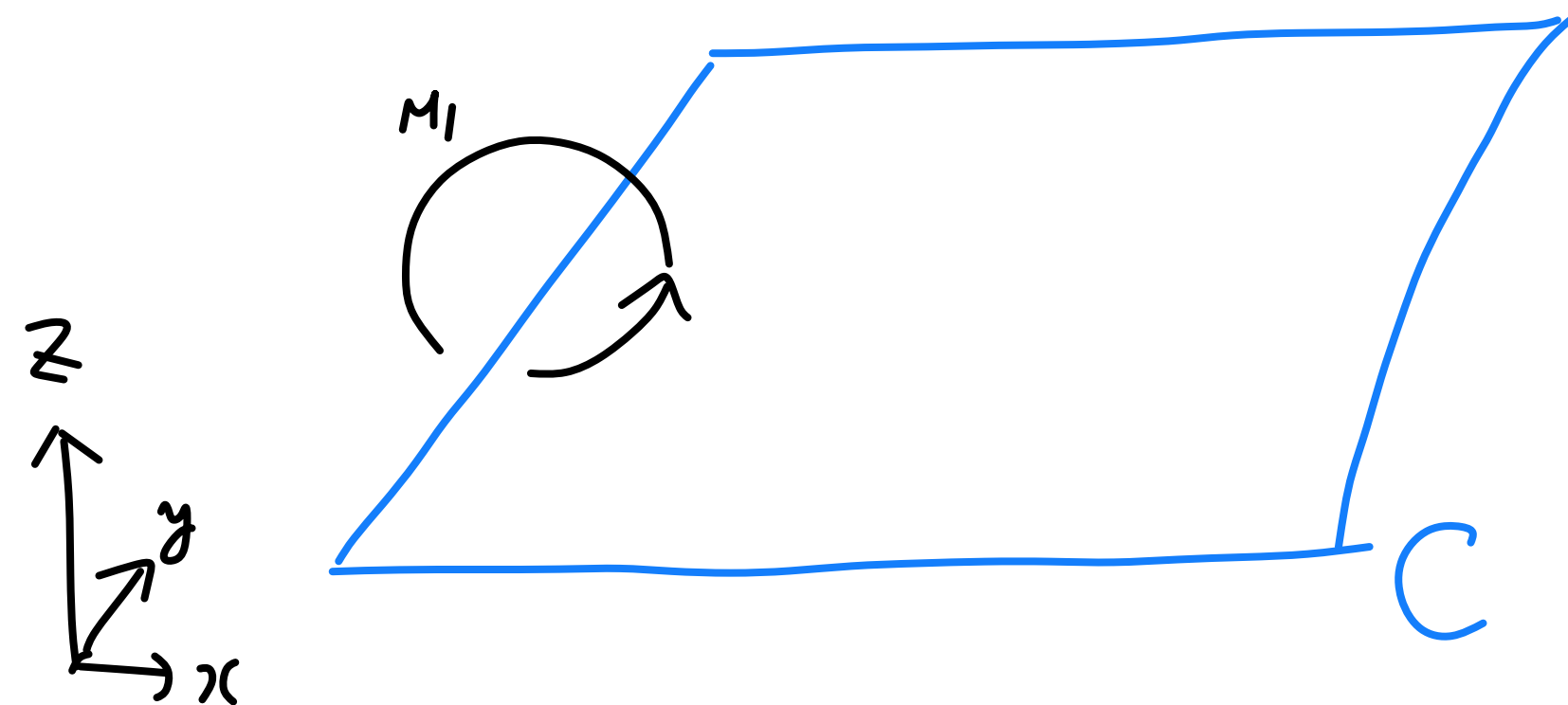
⇝ Only N-distinct strings can be naturally explained.

String tensions

In order to compute string tensions, we must know how the Wilson loops can be realized in the monopole theory

$$\mathcal{L}_{eff} = |d\vec{\sigma}|^2 + e^{-\frac{\#}{g^2}} \sum_{\alpha} \left(1 - \cos(\alpha \cdot \vec{\sigma})\right).$$
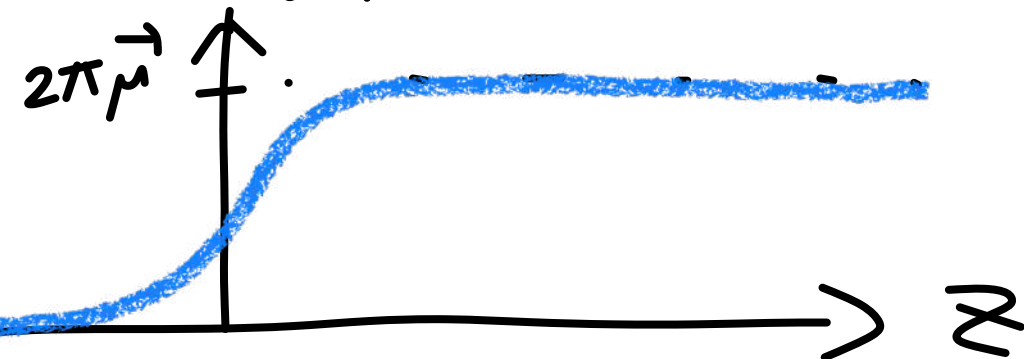
Defect operator

$$\oint_{M_1} d\vec{\sigma} = 2\pi \vec{\mu} \quad \left(\vec{\mu} \text{ is a weight vector}\right)$$

$$\Downarrow$$

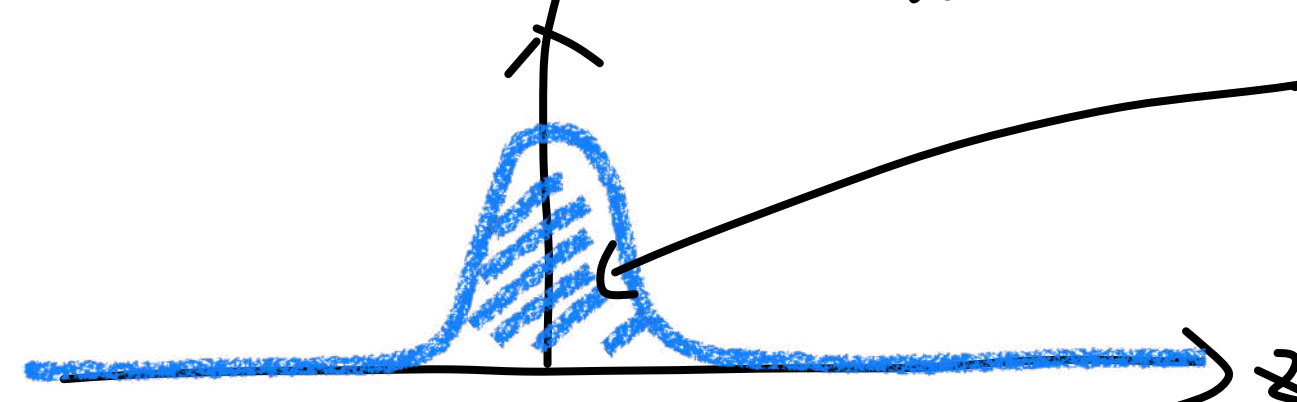$$W_{\vec{\mu}}(C) = e^{i \vec{\mu} \cdot \oint_C \vec{a}}.$$

σ - profile

Inside C, it looks like

Energy density

$$\varepsilon(z) = \mathcal{L}_{eff}(\sigma(z))$$

string tension $T_{\vec{\mu}}$.

String tensions beyond N-ality

Under a reasonable anatz, we obtain the string tensions for various charges:

$$\mu_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{1}{N} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad (SU(N) \text{ fundamental}. \quad N\text{-ality} = 1)$$

$$T_{\mu_1} \left(= \# e^{-\frac{\#}{g^2}} \times \frac{N-1}{\sqrt{N}} \right) \neq 0.$$

$$\mu_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \frac{2}{N} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad (SU(N) \text{ 2-index anti-sym}. \quad N\text{-ality} = 2)$$

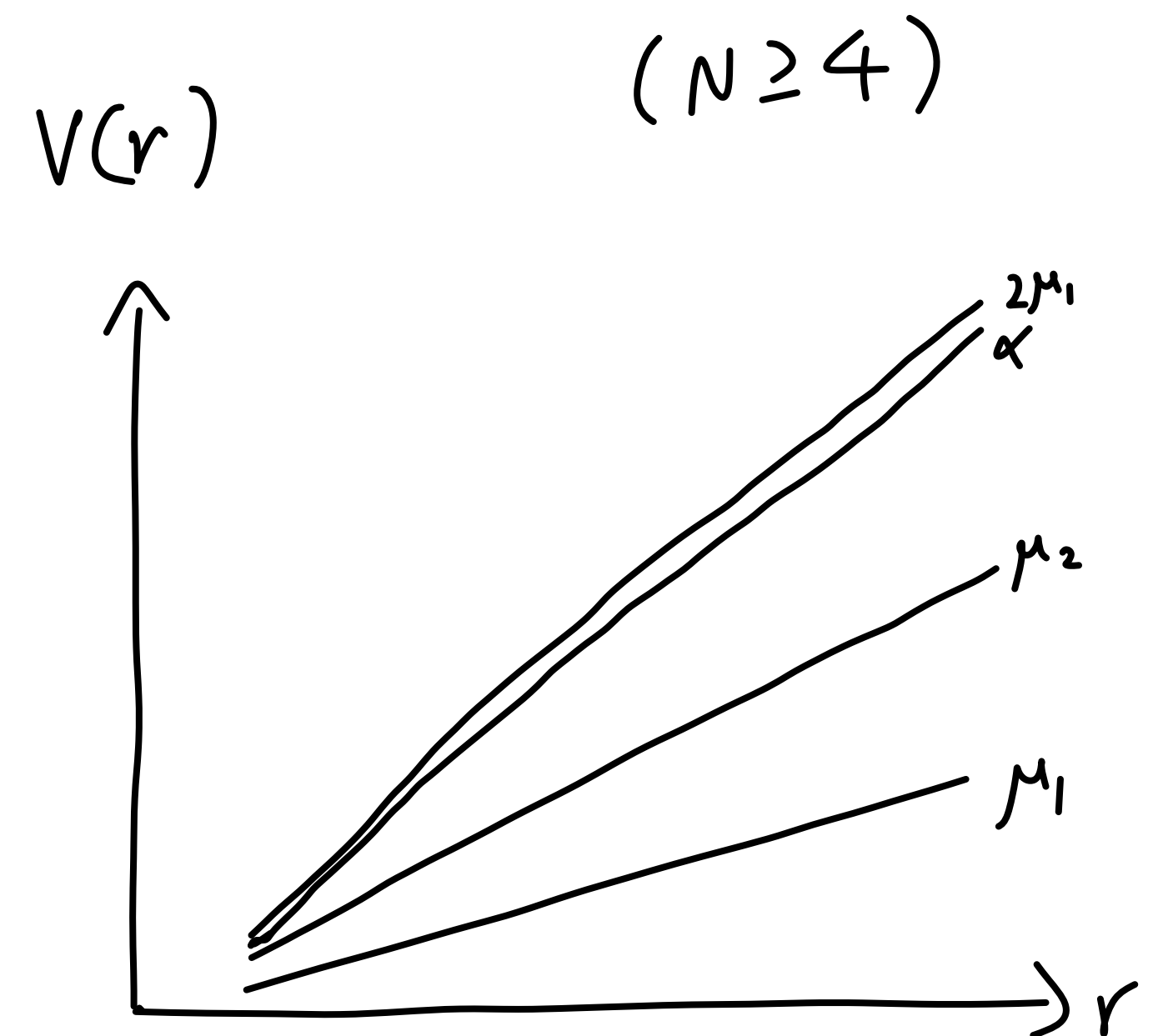$$T_{\mu_2} = \frac{2(N-2)}{N-1} T_{\mu_1} \qquad \left(< 2 T_{\mu_1} \right)$$

$$\underline{2\mu_1} \qquad (SU(N) \text{ 2-index sym}. \quad N\text{-ality} = 2)$$

$$T_{2\mu_1} = 2 T_{\mu_1}$$

$$\alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (SU(N) \text{ adjoint}. \quad N\text{-ality} = 0)$$

$$T_\alpha \simeq 2 T_{\mu_1}.$$

$$V(r) \qquad (N \geq 4)$$

Natural or unnatural?

For $U(1)^{N-1}$ model, this is natural. We have $U(1)^{N-1}$ 1-form sym. generated by

$$U_\theta^{(k)}(M_1) = \exp\left(i\frac{\theta}{2\pi}\oint_{M_1}\alpha_k\cdot d\vec{\sigma}\right) \qquad (k=1,\cdots,N-1).$$

$\Rightarrow$ <u>Infinitely many</u> strings can be explained by this symmetry.

For $U(1)^{N-1}\rtimes S_N$ model, these operators are not gauge invariant.

We must consider a special combination

$$U_n(M_1) = \prod_{k=1}^{N-1} U_{\frac{2\pi}{N}nk}^{(k)}(M_1)$$

$$= \exp\left(i\frac{n}{N}\oint_{M_1}(\alpha_1+2\alpha_2+\cdots+(N-1)\alpha_{N-1})\cdot d\sigma\right).$$

This obeys the $\mathbb{Z}_N$ multiplication law:

$$U_n U_m = U_{n+m \bmod N}.$$

$\Rightarrow$ Only N distinct strings can be expected. Unnatural situation ??

Non-invertible symmetry

We can construct another class of S_N-inv. operators from $U_\theta^{(k)}(M_1)$:

$$J_\theta(M_1) \equiv \frac{1}{N!} \sum_{P \in S_N} P \, U_\theta^{(1)}(M_1) \, P^{-1}$$

$$= \frac{1}{N(N-1)} \sum_{\alpha: \text{roots}} \exp\left(i \frac{\theta}{2\pi} \oint_{M_1} \alpha \cdot d\sigma\right).$$
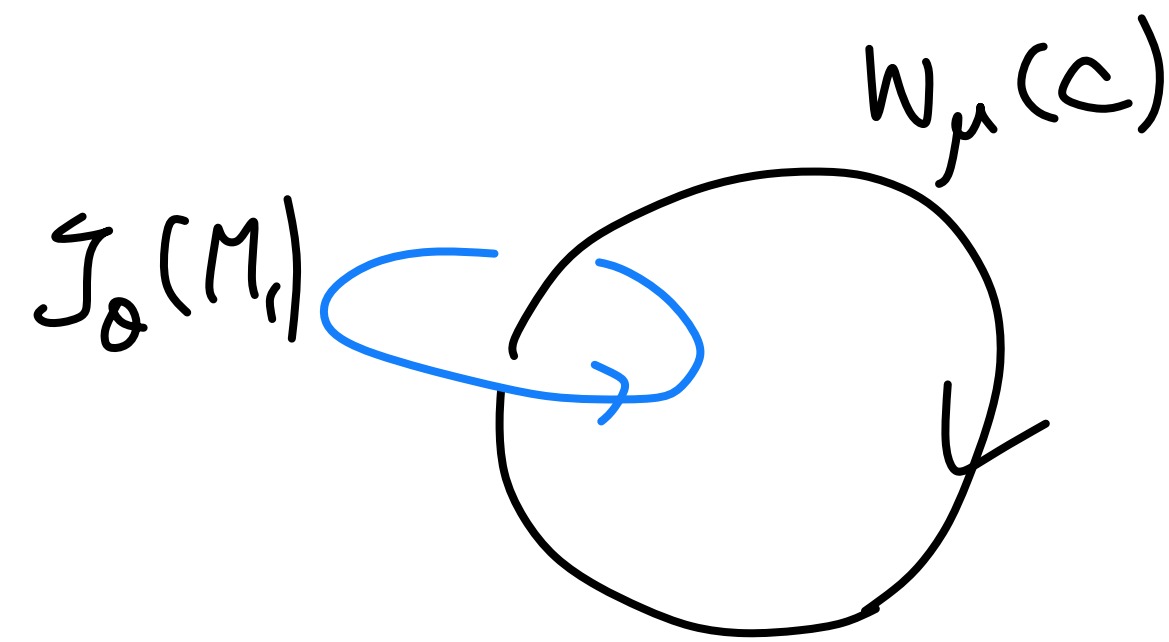
- $J_\theta(M_1)$ is topological $\Rightarrow$ "**Symmetry**"

$\Bigg\{$
- $J_\theta(M_1)$ does <u>not</u> form a group. $J_\theta \cdot J_{\theta'} \neq J_{\theta+\theta'}$

- Moreover, $J_\theta(M_1)$ does <u>not</u> correspond to unitary/anti-unitary operations.

Even though the last two properties are very unusual as symmetries, we regard J_θ as generators of "non-invertible symmetry".

(cf. Bhardwaj, Tachikawa '17, Buican, Gromov '17, Thorngren, Wang '19, Komargodski, Ohmori, Roumpedakis, Seifnashri '20 etc.)
(for 2d QFTs)

Transformation by $\mathcal{J}_\theta$

$$\mathcal{J}_\theta(M_1) \quad W_\mu(c) = \left(\frac{1}{N(N-1)} \sum_\alpha e^{i\theta \, \vec{\alpha} \cdot \vec{\mu}} \right) \times W_\mu(c)$$

$\underline{\mu = \mu_1}$ (N-ality 1)

$$W_{fd.} \longmapsto \frac{N - 2(1-\cos(\theta))}{N} \, W_{fd.}$$

As $\mathcal{J}_\theta$ is not unitary, its "eigenvalue" is smaller than $\underline{1}$.
Moreover, it can be 0 in some cases. ("non-invertible")

$\underline{\mu = \alpha_1}$ (Adjoint rep. N-ality 0)

$$W_{adj.} \longmapsto \frac{(N-2)(N-3) + 4(N-2)\cos(\theta) + 2\cos(2\theta)}{N(N-1)} \, W_{adj.}$$

$\rightsquigarrow$ $\mathcal{J}_\theta$ can distinguish the adj. rep. from the trivial rep.
Consistent with $T_\alpha \neq 0$.

N-ality 2 representations

As another exercise, we try to distinguish N-ality 2 representations,

2-index sym. ($\square\square$) and anti-sym ($\begin{array}{c}\square\\\square\end{array}$) reps.

highest weight $= 2\mu_1$ highest weight μ_2

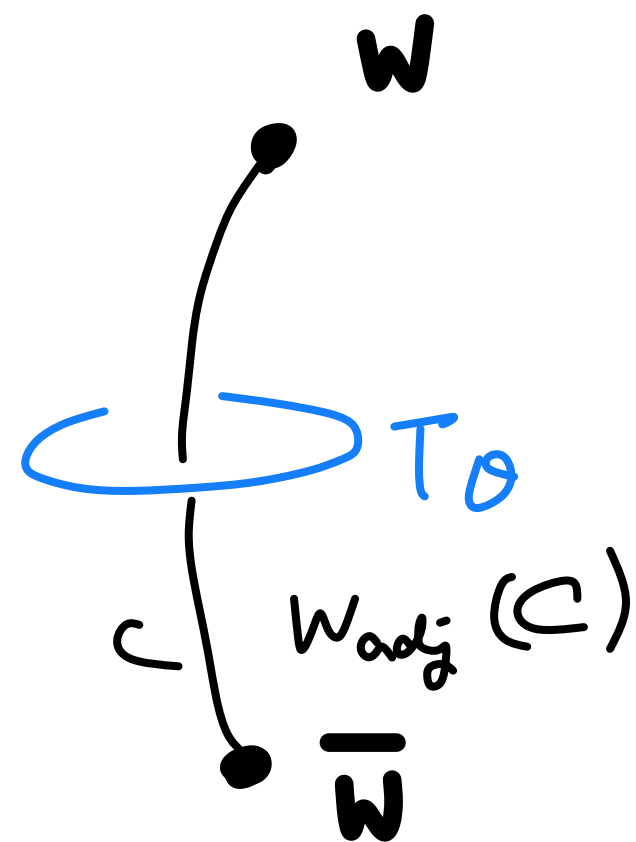Eigenoperators of $J_\theta(M_1)$ turn out to be W_{asym} and $(W_{sym} - W_{asym})$.

$$W_{sym} - W_{asym} \xmapsto{J_\theta} \frac{N - 2(1 - \cos(2\theta))}{N}(W_{sym} - W_{asym})$$

$$W_{asym} \xmapsto{J_\theta} \frac{(N-2)(N-3) + 2 + 4(N-2)\cos\theta}{N(N-1)} \cdot W_{asym}.$$

Consistent with $T_{\mu_2} \neq T_{2\mu_1}$.

Effect of dynamical charges

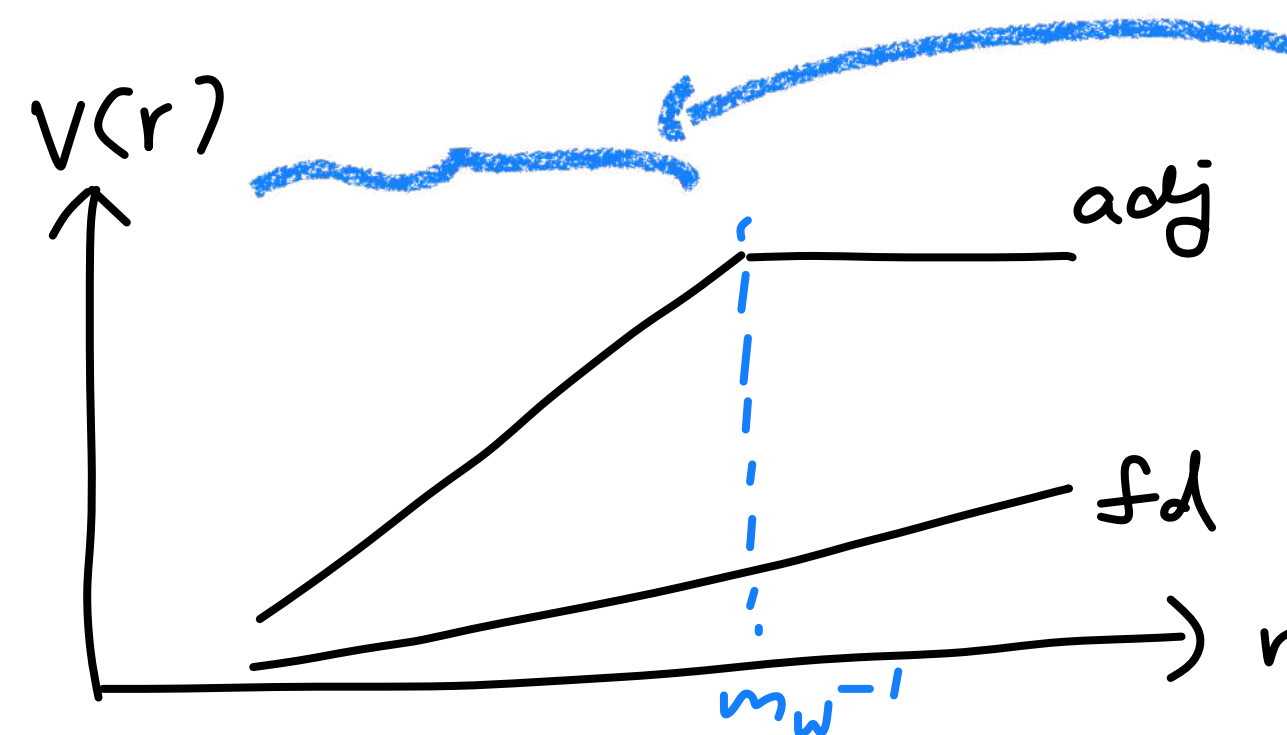Let us add dynamical electric matters with charges α's.

(W-bosons)

$\mathcal{T}_\theta$ is topological

$\Rightarrow \quad \mathcal{T}_\theta \cdot W_{adj} \overset{!}{=} W_{adj}.$

$\parallel$

$\dfrac{(N-2)(N-3) + 4(N-2)\cos\theta + 2\cos(2\theta)}{N(N-1)} W_{adj}$

The equation holds only if $\underline{\theta = 0}$ $(N \geq 3)$.

$\Rightarrow$ Non-invertible sym. is explicitly broken. (cf. Rudelius, Shao '20)

Still, if W's are heavy,

This regime can be understood by "softly-broken" non-invertible symmetry.

Speculations in Yang-Mills

In lattice simulation, the linear confinement is observed with the rep.-dep. string tensions.

(Casimir scaling, ...)

Also, in the large-N limit, factorization tells

$$\langle W_{adj}(C) \rangle \sim \langle W_{fd}(C) \rangle \cdot \langle W_{\overline{fd}}(C) \rangle$$

$$\Rightarrow \quad T_{adj} = 2 \, T_{fd} \, (\neq 0).$$

It's an interesting question if these behaviors could

be understandable by (approximate, or large-N emergent) non-invertible symmetries !!

Summary

① Construction of a toy model (3d semi-Abelian gauge theory)

- $G_{gauge} = U(1)^{N-1} \rtimes S_N$
- $\mathbb{Z}_N$ 1-form sym.

② Studying its string tensions (string tensions beyond N-ality)

- $T_{Adj} \simeq 2 T_{fd.} \neq 0$
- $U_1(M_1) = \exp\left(i \frac{1}{N} \oint_{M_1} (\alpha_1 + 2\alpha_2 + \cdots + (N-1)\alpha_{r-1}) \cdot d\sigma\right)$ detects N-ality.

③ "Symmetry" explanation on beyond N-ality (non-invertible sym. in 3d)

- $\mathcal{J}_\theta(M_1) = \frac{1}{N(N-1)} \sum_{\alpha: roots} \exp\left(i \frac{\theta}{2\pi} \int_{M_1} \alpha \cdot d\sigma\right)$

    is a topological, gauge-inv. operator.

- $\mathcal{J}_\theta \cdot W_{adj} = (\theta\text{-dep. factor}) \times W_{adj}.$

    $\Rightarrow$ Trivial N-ality can have nonzero string tensions.

Applicable to SU(N) YM ?